

# A Semidefinite Programming Framework for Controlling Multi–robot Systems in Dynamic Environments

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**Abstract**—In this paper, a discrete-time, semidefinite programming (SDP) framework is synthesized for controlling mobile robot teams operating in dynamic environments. Given an initially feasible configuration, the proffered framework embeds formation shape control and guarantees inter-agent and agent–obstacle collision avoidance and network interconnectivity across the formation given a sufficiently small  $\Delta t$  – provided that a feasible solution exists. Additionally, it affords goal-directed behaviors, which are explored, most notably, in terms of its application to directional coverage control, where the objective is to ensure that a set of mobile targets are being observed by at least a single member of the team at any given time. Central to our formulation is melding the recent application of shape theoretic constructs to globally optimal shape planning with state–dependent graphs whose enforced connectivity (gauged via their Fiedler value) implies satisfaction of the aforementioned constraints. Simulation results are presented to highlight the utility of our approach.

## I. INTRODUCTION

Robot teams collaborating to achieve some common objective can provide a level of robustness that often cannot be guaranteed by centralized, mono–agent systems. Unfortunately, there is an inherent tradeoff in considering such multi–agent systems. Their control is substantially more complex as each agent introduces additional degrees of freedom afforded by its mobility. Furthermore, they require teammates to effectively collaborate to bring about some desired system state. This implies algorithms that ensure some level of network connectivity or inter–agent visibility for communications. However, providing such performance guarantees for arbitrary systems operating in dynamic environments is still an open problem. It becomes inherently more complex when said environment is occupied with actors that can potentially exhibit competitive or unsafe behavior.

In an effort to address some of these key challenges, we leverage, extend, and integrate our previous results in optimal formation planning (see [1]) and omnidirectional coverage control (see [2]) to yield a rich, semidefinite framework for controlling autonomous robot teams. Specifically, we consider minimizing a convex functional subject to directional coverage, inter–agent and agent–obstacle proximity, and network interconnectivity constraints for teams in dynamic environments. Intuitively, the forthcoming results provide a fundamental theory uniting our previous work in [1] and [2].

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## II. RELATED WORK

Our focus is now directed towards recent results where formation, coverage, and/or network connectivity control was central to the primary results of the research. In [3], the authors synthesize feedback control laws for abstraction–based formation control in static environments subject to network connectivity constraints. In [4], the authors formulate decentralized controllers for swarms of robots to achieve a desired formation. In this work, the shape is implicitly defined as a contour of a smooth functional. Other related results appear in [5], [6], [7].

Coverage control has also garnered much attention in recent years. [8] develops distributed control laws based upon a centroidal Voronoi–tessellation coupled with a dynamic density function to enable simultaneous coverage and tracking. [9] utilizes virtual–bodies and artificial potential functions to formulate gradient–ascent control laws for mobile sensor networks. Additional work includes [10], [11], [12].

Others have explored network–based control as well. Most relevant to this research, however, is the work of [13], who considered a centralized algorithm for maximizing the connectivity of the communications graph by maximizing its state–dependent Laplacian’s Fiedler value. Subsequently, [14] showed that supergradient methods can be employed to yield a distributed solution. [15] also leverages this result.

In contrast to these prior efforts, our framework embeds a broader, holistic view of the control problem by considering formation control (where the notion of shape is fully captured without relaxation), directional (and, in fact, omnidirectional) coverage control, static and dynamic collision avoidance, network connectivity, all while still attempting to minimize an additional, arbitrary convex objective.

## III. PROBLEM STATEMENT

Begin by letting  $\mathcal{R} = \{r_1, \dots, r_n\}$  denote a finite set of  $n$  mobile robots with time–dependent state vector  $q^{\mathcal{R}}(t) = (q_1^{\mathcal{R}}(t)^T, \dots, q_n^{\mathcal{R}}(t)^T)^T \in \mathbb{R}^{2n}$ ,  $q_i^{\mathcal{R}}(t) = (q_{i,x}^{\mathcal{R}}(t), q_{i,y}^{\mathcal{R}}(t))^T \in \mathbb{R}^2$  denoting the system’s state at time  $t$  with respect to inertial frame  $\mathcal{F}_{\mathcal{W}}$ . Additionally, assume

$$\dot{q}_i^{\mathcal{R}} = u_i^{\mathcal{R}} \in \mathcal{U}^{\mathcal{R}} \quad (1)$$

with each team member having maximum translational velocity  $v_{i,\tau}^{\mathcal{R}} \in \mathbb{R}$ ,  $v_{i,\tau}^{\mathcal{R}} \geq 0$ .

As our objective, in part, is to embed dynamic collision–avoidance, let  $\mathcal{O} = \{o_1, \dots, o_m\}$ ,  $m \in \mathbb{Z}$ ,  $m \geq 0$  denote a finite set of obstacles having dynamics (1) with maximal translational velocity  $v_{j,\tau}^{\mathcal{O}}$  similarly defined as  $v_{i,\tau}^{\mathcal{R}}$ . Accordingly, associate with  $\mathcal{O}$  the time–dependent state vector  $q^{\mathcal{O}}(t) = (q_1^{\mathcal{O}}(t)^T, \dots, q_m^{\mathcal{O}}(t)^T)^T$ ,  $q_j^{\mathcal{O}} \in \mathbb{R}^2$ . For notational convenience, define  $q^{\mathcal{R}} \triangleq q^{\mathcal{R}}(t)$  and  $q_i^{\mathcal{R}} \triangleq q_i^{\mathcal{R}}(t)$  and adopt

similar definitions for  $q^{\mathcal{O}}$  and  $q_j^{\mathcal{O}}$ . Additionally, it is assumed without loss of generality that  $\mathcal{R} \cap \mathcal{O} = \emptyset$ .

Given the definitions of  $\mathcal{R}$  and  $\mathcal{O}$ , the following assumptions can now be formally stated

- A1**  $\forall r_i \in \mathcal{R}$ ,  $r_i$  has a radially symmetric communication broadcast range  $b_i^{\mathcal{R}} \in \mathbb{R}^+$
- A2**  $\forall r_i \in \mathcal{R}$ ,  $r_i$  has an omnidirectional sensing modality (e.g., omnidirectional LIDAR) of radius  $d_i^{\mathcal{R}} \in \mathbb{R}^+$  that allows it to estimate the *relative pose* and *velocity* of objects within  $d_i^{\mathcal{R}}$  units of  $q_i^{\mathcal{R}}$  (to facilitate collision-avoidance).
- A3**  $\forall r_i \in \mathcal{R}$ ,  $r_i$  has a minimal proximity tolerance  $p_i^{\mathcal{R}} \in \mathbb{R}^+$  defined such that  $p_i^{\mathcal{R}} < \{d_i^{\mathcal{R}}, b_i^{\mathcal{R}}\}$
- A4**  $\forall o_j \in \mathcal{O}$ ,  $o_j$  has a minimal proximity tolerance  $p_j^{\mathcal{O}} \in \mathbb{R}^+$  and a unique identifier known by  $\forall r_i \in \mathcal{R}$ ,  $\|q_i^{\mathcal{R}} - q_j^{\mathcal{O}}\|_2 \leq d_i^{\mathcal{R}}$

In addition to **A1 – A4**, our problem involves an additional assumption to facilitate formation shape control

- A5** Assume that the team has knowledge of a desired shape formation given by the *shape vector*  $q^{\mathcal{S}} = (q_1^{\mathcal{S}}, \dots, q_n^{\mathcal{S}})^T \in \mathbb{R}^{2n}$  defined w.r.t. shape frame  $\mathcal{F}_{\mathcal{S}}$ .

Momentarily, details of the chosen shape representation are deferred until §IV-A with the exception of noting that the team would like to achieve an equivalent representation of  $q^{\mathcal{S}}$  (up to a Euclidean similarity transformation) in  $\mathcal{F}_{\mathcal{W}}$ .

Given these definitions and assumptions, this research is motivated by the following key problem

*Problem 1:* Given the sets  $\mathcal{R}$  and  $\mathcal{O}$  with dynamics given by (1) and states  $q^{\mathcal{R}} \in \mathbb{R}^n$  and  $q^{\mathcal{O}} \in \mathbb{R}^m$  defined with respect to  $\mathcal{F}_{\mathcal{W}}$  and assumptions **A1–A5**, can a general control framework be devised such that

- 1) Given relative shape geometry  $q^{\mathcal{S}} \in \mathcal{F}_{\mathcal{S}}$ , the team is driven to achieve an equivalent shape in  $\mathcal{F}_{\mathcal{W}}$
- 2) Proximity constraints are satisfied *vt. i.e.*

$$\forall r_i \in \mathcal{R}, \forall o_j \in \mathcal{O}, \|q_i^{\mathcal{R}} - q_j^{\mathcal{O}}\|_2 > \max\{p_i^{\mathcal{R}}, p_j^{\mathcal{O}}\}$$

$$\forall r_i, r_j \in \mathcal{R}, i \neq j, \|q_i^{\mathcal{R}} - q_j^{\mathcal{R}}\|_2 > \max\{p_i^{\mathcal{R}}, p_j^{\mathcal{R}}\}$$

- 3) *vt*, Network connectivity remains intact

## IV. INTERNAL FORMATION CONTROL

Our approach in formulating our framework is “inside-out” in the sense that our initial formulation focuses upon internal formation control (with respect to shape and inter-agent collision avoidance) where the influence of external agents and obstacles are momentarily ignored. Accordingly, we begin by considering ensemble shape control.

### A. Embedding Ensemble Shape Control

*Shape* is defined differently in different contexts (e.g., see [5], [6]). However, for the purposes of this research, we follow the approach of [1] and adopt a shape theoretic definition (see [16]), which states that two formation shapes are equivalent if and only if there is Euclidean similarity transformation mapping one into the other. Adopting standard convention, we let  $q \sim q^{\mathcal{S}}$  denote an equivalence

relation (*i.e.*,  $q \sim q^{\mathcal{S}} \iff q \in [q^{\mathcal{S}}]$ ) with  $[q^{\mathcal{S}}]$  denoting the equivalence class of shapes giving  $q^{\mathcal{S}}$ .

Given this definition, consider the following result of [1]

*Theorem 4.1:* Let  $q \in \mathbb{R}^n$  be defined with respect to world frame  $\mathcal{F}_{\mathcal{W}}$  and let  $q^{\mathcal{S}} \in \mathbb{R}^n$  denote the desired shape geometry defined with respect to shape frame  $\mathcal{F}_{\mathcal{S}}$ . Assume (without loss of generality) that  $q_i$  is assigned relative shape coordinate  $q_i^{\mathcal{S}}$  and  $q_1^{\mathcal{S}} \triangleq O_{\mathcal{F}_{\mathcal{S}}}$  in the formation, then

$$q \sim q^{\mathcal{S}} \iff q \in \ker A_{\mathcal{S}} \quad (2)$$

where  $A_s \in \mathbb{R}^{2(n-2) \times n}$  is the *linear shape operator* given by the following linear transformation on  $q$

$$\begin{aligned} \|q_i^{\mathcal{S}} - q_1^{\mathcal{S}}\|_2 (q_{i,x} - q_{1,x}) &= (q_{i,x}^{\mathcal{S}}, -q_{i,y}^{\mathcal{S}})^T (q_2 - q_1) \\ \|q_i^{\mathcal{S}} - q_1^{\mathcal{S}}\|_2 (q_{i,y} - q_{1,y}) &= (q_{i,y}^{\mathcal{S}}, q_{i,x}^{\mathcal{S}})^T (q_2 - q_1) \end{aligned} \quad (3)$$

Although [1] exploits this result to embed *homogenous shape constraints* into convex optimization problems, our application of Theorem 4.1, here, is slightly different. Specifically, our objective is to obtain some element  $q \in \ker A_{\mathcal{S}}$  (*i.e.*, a feasible shape formation). A natural approach to this end is to seek inspiration from dynamical systems and consider the asymptotically stable, feedback control policy  $\dot{q}^{\mathcal{R}} = -A_{\mathcal{S}}^T A_{\mathcal{S}} q^{\mathcal{R}}$ . Although elegant, it is unclear how such a controller can be augmented to provide substantial performance guarantees such as maintenance of network interconnectivity and static collision-avoidance.

Our attention instead turns towards the application of convex optimization. Begin by defining the positive semidefinite, *shape-disparity* function  $f_{\mathcal{S}}(q^{\mathcal{R}}) : \mathbb{R}^n \rightarrow \mathbb{R}^+ \cup \{0\}$  given by

$$f_{\mathcal{S}}(q^{\mathcal{R}}) = q^{\mathcal{R}^T} A_{\mathcal{S}}^T A_{\mathcal{S}} q^{\mathcal{R}} = \|A_{\mathcal{S}} q^{\mathcal{R}}\|_2^2 \quad (4)$$

Observe that  $f_{\mathcal{S}}$  is convex and features a global minimum at  $f_{\mathcal{S}}(q^{\mathcal{R}}) = 0$ , which serves to indicate  $q^{\mathcal{R}} \sim q^{\mathcal{S}}$ .

### B. Ensuring Inter-agent Collision Avoidance

To embed inter-agent collision-avoidance, consider the inter-agent proximity graph

*Definition 1:* The *Inter-agent Proximity Graph* is a weighted, simple graph  $G_P^{\mathcal{R}}(\mathcal{V}_P^{\mathcal{R}}, \mathcal{E}_P^{\mathcal{R}})$  with  $\mathcal{V}_P^{\mathcal{R}} = \mathcal{R}$  and edges given by the set  $\mathcal{E}_P^{\mathcal{R}} = \mathcal{R} \times \mathcal{R} \setminus \{(r_i, r_i) : r_i \in \mathcal{R}\}$  with weights given by  $f_P^{\mathcal{R}}$  as follows

$$f_P^{\mathcal{R}}(r_i, r_j) = \begin{cases} 0, & \|q_i^{\mathcal{R}} - q_j^{\mathcal{R}}\|_2 \leq \max\{p_i^{\mathcal{R}}, p_j^{\mathcal{R}}\} \\ 1, & \text{otherwise} \end{cases} \quad (5)$$

It is clear that if  $G_P^{\mathcal{R}}$  is *complete* (*i.e.*,  $G_P^{\mathcal{R}} = K_n$ ) that all proximity constraints are satisfied. Accordingly, we would like to enforce completeness with respect to time. Consider a state-dependent variation of  $G_P^{\mathcal{R}}$  where the indicator  $f_p^{\mathcal{R}}$  is supplanted with a smooth continuous functional that captures its behavior. Such a formulation would facilitate applying first-order discretization techniques as in [13].

One approach is to exploit a standard sigmoidal function and define it such that its inflection point lies at the chosen proximity threshold. More specifically, we abuse notation, and redefine  $f_P^{\mathcal{R}}$  as follows

$$f_P^{\mathcal{R}}(q_i^{\mathcal{R}}, q_j^{\mathcal{R}}) = \left(1 + e^{\gamma_P^{\mathcal{R}} (\max\{p_i^{\mathcal{R}}, p_j^{\mathcal{R}}\} - \|q_i^{\mathcal{R}} - q_j^{\mathcal{R}}\|_2)}\right)^{-1} \quad (6)$$

Figure 1 illustrates this functional. Observe that as  $\gamma_P^R \rightarrow \infty$  the function behaves as an indicator.

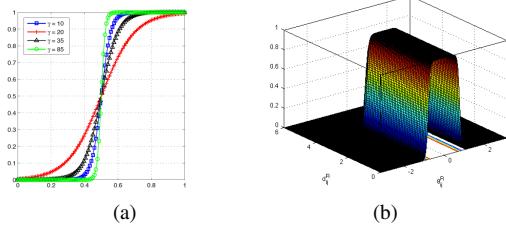


Fig. 1. (a) Chosen sigmoid function for  $f_P^R$  with varying  $\gamma_P$ . (b) A product of sigmoids can be used to enforce multi-dimensional constraints such as those associated with directional visibility (see §VI)

In addition to being smooth,  $f_P^R$  is now a direct functional of the time-dependent state vectors  $q^R$ ; however, before proceeding with formal analysis, recall the classical results from algebraic graph theory [17] which states that the second-smallest eigenvalue (*i.e.*, its Fiedler value, which shall be denoted  $\lambda_2(G)$ ) of the weighted graph Laplacian for a connected graph  $G$  is a direct measure of the underlying interconnectivity of the graph.

For  $G_P^R$ , the weighted graph Laplacian is given as follows

$$[\mathcal{L}_P^R]_{uv} = \begin{cases} \sum_{s \neq v} f_P^R(r_u, r_s), u = v \\ -f_P^R(r_u, r_v), u \neq v \end{cases} \quad (7)$$

Recalling that  $f_P^R$  is state-dependent, it follows that so too is its Fiedler value – *i.e.*,  $\lambda_2(G_P)$  (see [13]). As such, if  $\lambda_2(G_P)$  can be constrained to its maximal value with respect to time, given an initially feasible configuration (*i.e.*, state-dependent  $G_P^R = K_n$ ), it follows that the team will always choose collision-free trajectories provided that they exist.

Such a constraint is non-convex and, thus, does not lend itself to efficient optimization techniques. In order to address this issue, our approach follows that of [13], who apply a first-order discretization techniques to yield a discrete-time, semidefinite representation.

Following suit, we discretize  $f_P^R$  to yield

$$\Delta f_P^R = \frac{\gamma_P^R e^{\gamma_P^R(p_{ij}^R - d_{ij}^R)} (q_i^R(k) - q_j^R(k))^T (\Delta q_i^R - \Delta q_j^R)}{\left[ d_{ij}^R \left( 1 + e^{\gamma_P^R(p_{ij}^R - d_{ij}^R)} \right) \right]^2} \quad (8)$$

where  $p_{ij}^R = \max\{p_i^R, p_j^R\}$ ,  $d_{ij}^R = \|q_i^R - q_j^R\|_2$ ,  $\Delta q_i^R = f_P^R(k+1) - f_P^R(k)$  and  $\Delta q_j^R = q_i^R(k+1) - q_i^R(k)$ .

Given Equation (8),  $\mathcal{L}_P^R$  lends itself to a discrete-time form given as follows

$$[\Delta \mathcal{L}_P^R]_{uv} = \begin{cases} \sum_{s \neq v} \Delta f_P^R, u = v \\ -\Delta f_P^R, u \neq v \end{cases} \quad (9)$$

where  $\Delta \mathcal{L}_P^R = \mathcal{L}_P^R(k+1) - \mathcal{L}_P^R(k)$ . In this formulation,  $\mathcal{L}_P^R(k+1)$  is now a linear functional of  $\mathcal{L}_P^R(k)$  and  $\Delta \mathcal{L}_P^R$ , where the latter is computed given the state estimate  $q^R(k)$ .

Given this result and recalling that the upper bound for  $\lambda_2(G_P^R)$  is given by  $|\mathcal{E}_P^R|$  (see [17]), enforcing completeness over  $\Delta t$  reduces to the following

$$J_P^R{}^T \mathcal{L}_P^R(k+1) J_P^R = n I_{n-1} \quad (10)$$

where  $J_P^R \in \mathbb{R}^{n-1 \times n}$  is an orthonormal basis such that  $\forall q \in \text{span } J_P^R, 1^T q = 0$ , with  $1$  again denoting the vector of all 1's and  $I_{n-1}$  denoting the  $n-1$  dimensional identity matrix (see [13] for details regarding this sort of projection).

Given this equality and some sufficiently small epoch,  $\Delta t$ , our semidefinite framework has begun to take form as (4) provides our desired, discrete-time objective, while (10) provides a means for guaranteeing inter-agent collision avoidance. This result naturally follows from the intuitive relationship  $\Delta t \rightarrow 0 \Rightarrow \Delta q^R \rightarrow 0$ .

### C. Ensuring Network Interconnectivity

As enforcing network connectivity constraints via a state-dependent network graph Laplacian has been previously explored (see [2]), only a brief formulation of the approach employed to yield our results is presented.

Let  $G_N(\mathcal{V}_N, \mathcal{E}_N)$  be a weighted, simple graph with  $\mathcal{V}_N = \mathcal{R}$  and  $\mathcal{E}_N = \mathcal{R} \times \mathcal{R} \setminus \{(r_i, r_i) : r_i \in \mathcal{R}\}$  with weight function  $f_N^R$  serving as a discrete-indicator of connectivity. For the purposes of this research,  $f_N^R$  is defined

$$f_N^R(q_i^R, q_j^R) = \left( 1 + e^{\gamma_N(\|q_i^R - q_j^R\|_2 - \min\{b_i^R, b_j^R\})} \right)^{-1} \quad (11)$$

Adopting the approach of §IV-B yields a discrete form similar to (8) with respect to  $\gamma_N$ ,  $b_{ij}^R = \min\{b_i^R, b_j^R\}$ , and exponent  $\gamma_N(d_{ij}^R - b_{ij}^R)$ . From this result, network connectivity over  $\Delta t$  can be enforced via the linear matrix inequality

$$J_N^T \mathcal{L}_N(k+1) J_N \succ \epsilon_N I_{n-1} \quad (12)$$

with  $0 < \epsilon_N \leq n = |\mathcal{E}_N| = |\mathcal{R}|$  representing the desired level of connectivity to be maintained across the formation.

### D. An Internal Formation Control SDP Framework

Given this initial discussion, a discrete-time SDP framework can now be formulated that aims to minimize shape disparity and guarantees inter-agent collision avoidance and network interconnectivity across the formation. This is formally stated in the following theorem, whose proof follows trivially from the preceding discussion and analysis

**Theorem 4.2:** Let  $\mathcal{R}$  be as previously defined in accordance with **A1 – A5** with elements having dynamics (1). Let  $\Gamma: \mathbb{R}^{2n} \rightarrow \mathbb{R}$  be a convex objective to be minimized over a sufficiently small  $\Delta t$ . Solving the following SDP at step  $k$

$$\begin{aligned} & \min \Gamma(q^R(k+1)) + \gamma_s z_s(k+1) \\ & \text{s.t. } \|q_i^R(k+1) - q_i^R(k)\|_2 \leq v_{i,\tau}^R \Delta t, \quad i = 1, \dots, n \\ & \quad \|A_S q^R(k+1)\|_2 \leq z_s(k+1) \\ & \quad J_P^R{}^T \mathcal{L}_P^R(k+1) J_P^R = n I_{n-1} \\ & \quad J_N^T \mathcal{L}_N(k+1) J_N \succ \epsilon_N I_{n-1} \end{aligned} \quad (13)$$

will result in trajectories that minimize  $\Gamma(q^R(k+1)) + \gamma_s z_s(k+1)$  over  $\Delta t$ , while guaranteeing network connectivity and collision-free trajectories – provided that the team begins in an initially feasible configuration<sup>1</sup>.

<sup>1</sup>Although not explicitly stated, from this point forward initial feasibility should be understood as a precondition for each of the forthcoming results.

In this formulation,  $n$  second-order conic inequalities have been introduced to serve as velocity constraints and to mitigate the effects of the linearization. Additionally, the objective has an additional summand  $\Gamma: \mathbb{R}^{2n} \rightarrow \mathbb{R}$  representing an arbitrary convex functional that the team should attempt to minimize. The terms  $\gamma_s \in \mathbb{R}$ ,  $\gamma_s \geq 0$  serves as a gain, which can be utilized to enable, disable, or assign priority to the shape disparity functional over  $\Gamma(q^{\mathcal{R}})$ .

It should be emphasized that Theorem 4.2 *does not* imply convergence of the chosen objectives to a global optimum. Nevertheless, it *does guarantee* that the team will minimize these objectives over  $\Delta t$ , while guaranteeing both network interconnectivity and inter-agent collision avoidance<sup>2</sup>.

## V. EXTERNAL FORMATION CONTROL

Given Theorem 4.2, our focus now turns to addressing the influence of external actors on the team's motions.

### A. Reactive Collision Avoidance for Dynamic Obstacles

Begin by considering the following extension to the Inter-agent Proximity Graph

**Definition 2:** The Proximity Graph is the weighted, simple graph  $G_P(\mathcal{V}_P, \mathcal{E}_P)$  with  $\mathcal{V}_P = \mathcal{R} \cup \mathcal{O}$  and edges given by the set  $\mathcal{E}_P = (\mathcal{R} \cup \mathcal{O}) \times (\mathcal{R} \cup \mathcal{O}) \setminus \{(r_i, r_i) \cup (o_j, o_j) : r_i \in \mathcal{R}, o_j \in \mathcal{O}\}$  with weights

$$f_P(y_i, y_j) = \begin{cases} f_P^{\mathcal{R}}(y_i, y_j), & (y_i, y_j) \in \mathcal{R} \times \mathcal{R} \\ f_P^{\mathcal{O}}(y_i, y_j), & (y_i, y_j) \in \mathcal{R} \times \mathcal{O} \\ 1, & (y_i, y_j) \in \mathcal{O} \times \mathcal{O} \end{cases} \quad (14)$$

where  $f_P^{\mathcal{R}}$  is as previously defined in (6) and  $f_P^{\mathcal{O}}$  is similarly defined as follows

$$f_P^{\mathcal{O}}(q_i^{\mathcal{R}}, q_j^{\mathcal{O}}) = \left(1 + e^{\gamma_P(\max\{p_i^{\mathcal{R}}, p_j^{\mathcal{O}}\} - \|q_i^{\mathcal{R}} - q_j^{\mathcal{O}}\|_2)}\right)^{-1} \quad (15)$$

Observe that if  $\mathcal{O} = \emptyset$  then the general proximity graph simplifies to the inter-agent proximity graph. In other words,  $G_P^{\mathcal{R}} \subseteq G_P$ . Similarly, if  $G_P = K_{n+m}$  then all proximity constraints (both inter-agent and agent-obstacle) are satisfied. Discretizing as in §IV-B, this constraint can be articulated as the following linear matrix equality

$$J_P^T \mathcal{L}_P(k+1) J_P = (n+m-1) I_{n+m-1} \quad (16)$$

where  $[\Delta \mathcal{L}_P]_{uv}$  is defined similarly to (9) with

$$\Delta f_P(y_i, y_j) = \begin{cases} \Delta f_P^{\mathcal{R}}(y_i, y_j), & (y_i, y_j) \in \mathcal{R} \times \mathcal{R} \\ \Delta f_P^{\mathcal{O}}(y_i, y_j), & (y_i, y_j) \in \mathcal{R} \times \mathcal{O} \\ 0, & (y_i, y_j) \in \mathcal{O} \times \mathcal{O} \end{cases} \quad (17)$$

and  $\Delta f_P^{\mathcal{R}}$  defined as in (8) and  $\Delta f_P^{\mathcal{O}}$  being given by

$$\Delta f_P^{\mathcal{O}} = \frac{\gamma_P e^{\gamma_P(p_{ij}^{\mathcal{O}} - d_{ij}^{\mathcal{O}})}}{[d_{ij}^{\mathcal{R}} \left(1 + e^{\gamma_P(p_{ij}^{\mathcal{O}} - d_{ij}^{\mathcal{O}})}\right)]^2} (q_i^{\mathcal{R}}(k) - q_j^{\mathcal{O}}(k))^T (\Delta q_i^{\mathcal{R}} - \Delta q_j^{\mathcal{O}}) \quad (18)$$

where  $p_{ij}^{\mathcal{O}} = \max\{p_i^{\mathcal{R}}, p_j^{\mathcal{O}}\}$ ,  $d_{ij}^{\mathcal{O}} = \|q_i^{\mathcal{R}} - q_j^{\mathcal{O}}\|_2$ ,  $\Delta f_P^{\mathcal{O}} = f_P^{\mathcal{O}}(k+1) - f_P^{\mathcal{O}}(k)$ , and  $\Delta q_j^{\mathcal{O}}$  corresponds to the team's discrete-time estimate of  $o_j$ 's velocity at time  $k$ .

<sup>2</sup>Similar guarantees apply to the forthcoming Theorems/Corollaries

## B. A General SDP Control Framework

Given this analysis and our previous discussion, the following result trivially follows

**Theorem 5.1:** Let  $\mathcal{R}$  and  $\mathcal{O}$  be as previously defined in accordance with **A1 – A5** with elements having dynamics (1). Let  $\Gamma: \mathbb{R}^{2n} \rightarrow \mathbb{R}$  and  $\Delta t$  be defined as in Theorem 4.2. Solving the following SDP at step  $k$

$$\begin{aligned} \min \quad & \Gamma(q^{\mathcal{R}}(k+1)) + \gamma_s z_s(k+1) \\ \text{s.t.} \quad & \|q_i^{\mathcal{R}}(k+1) - q_i^{\mathcal{R}}(k)\|_2 \leq v_{i,\tau}^{\mathcal{R}} \Delta t, \quad i = 1, \dots, n \\ & \|A_S q^{\mathcal{R}}(k+1)\|_2 \leq z_s(k+1) \\ & J_P^T \mathcal{L}_P(k+1) J_P = (n+m) I_{n+m-1} \\ & J_N^T \mathcal{L}_N(k+1) J_N \succ \epsilon_N I_{n-1} \end{aligned} \quad (19)$$

will result in trajectories that minimize  $\Gamma(q^{\mathcal{R}}(k+1)) + \gamma_s z_s(k+1)$  over  $\Delta t$ , while guaranteeing network connectivity and collision-free trajectories (between agents and obstacles) – provided that a feasible solution exists.

Although not formally stated, a straightforward corollary to Theorem 5.1 is that *in static environments, inter-agent collisions and agent-obstacle collisions are guaranteed for a sufficiently small  $\Delta t$* .

## VI. EMBEDDING GOAL-DIRECTED BEHAVIORS

Our attention now turns towards extending these results to explicitly include goal-directed behaviors.

### A. Abstraction-based Waypoint Traversal

Towards this end, we augment **A1–A5** with assumption **A6**  $\mathcal{R}$  has been supplied an objective waypoint  $q^{\mathcal{G}} \in \mathbb{R}^2$  with respect to  $\mathcal{F}_{\mathcal{W}}$ .

Consider the *virtual-leader* abstraction whose time-dependent state is given by  $q_{\mu}^{\mathcal{R}}$ , which is a linear combination of the team's state variables  $q_i^{\mathcal{R}}$

$$q_{\mu}^{\mathcal{R}} = (w^T 1)^{-1} w^T q^{\mathcal{R}} \quad (20)$$

In this formulation,  $1 \in \mathbb{R}^n$  denotes the vector of all 1's and  $w = (w_1, \dots, w_n)^T \in \mathbb{R}^n$  is a vector of weights representing the influence of agent  $r_i$ 's state on the virtual state  $q_{\mu}^{\mathcal{R}}$ .

Given  $q_{\mu}^{\mathcal{R}}$ , our approach considers

$$f_{\mathcal{G}}(q_{\mu}^{\mathcal{R}}, q^{\mathcal{G}}) = \|q_{\mu}^{\mathcal{R}} - q^{\mathcal{G}}\|_2 \quad (21)$$

to capture the team's disparity from  $q^{\mathcal{G}}$ . Intuitively, the team achieves its goal when the virtual-leader achieves  $q^{\mathcal{G}}$ .

Accordingly, the following corollary is presented, whose proof is omitted as it trivially follows from our discussion

**Corollary 6.1:** Let  $\mathcal{R}$  and  $\mathcal{O}$  be as previously defined in accordance with **A1 – A6** with elements having dynamics (1). Let  $\Gamma(q^{\mathcal{R}}): \mathbb{R}^{2n} \rightarrow \mathbb{R}$  and  $\Delta t$  be defined as in Theorem 4.2. Solving the following SDP at step  $k$

$$\begin{aligned} \min \quad & \Gamma(q^{\mathcal{R}}(k+1)) + \gamma_s z_s(k+1) + \gamma_g z_g(k+1) \\ \text{s.t.} \quad & \|q_i^{\mathcal{R}}(k+1) - q_i^{\mathcal{R}}(k)\|_2 \leq v_{i,\tau}^{\mathcal{R}} \Delta t, \quad i = 1, \dots, n \\ & \|A_S q^{\mathcal{R}}(k+1)\|_2 \leq z_s(k+1) \\ & \|w^T q^{\mathcal{R}}(k+1) - (1^T w) q^{\mathcal{G}}\|_2 \leq z_g(k+1) \\ & J_P^T \mathcal{L}_P(k+1) J_P = (n+m) I_{n+m-1} \\ & J_N^T \mathcal{L}_N(k+1) J_N \succ \epsilon_N I_{n-1} \end{aligned} \quad (22)$$

where  $\gamma_g \geq 0$ , will result in trajectories that minimize  $\Gamma(q^{\mathcal{R}}(k+1)) + \gamma_s z_s(k+1) + \gamma_g z_g(k+1)$  over  $\Delta t$ , while guaranteeing network connectivity and collision-free trajectories – provided that a feasible solution exists.

### B. Directional Coverage Control for Target Tracking

We now demonstrate goal-directed behavior in terms of constraint-satisfaction by extending [2]. Redefine  $q^{\mathcal{R}}$  and  $q^{\mathcal{O}}$  such that  $q_i^{\mathcal{R}}, q_j^{\mathcal{O}} \in SE(2)$ . In other words,  $q_i^{\mathcal{R}} = (q_{i,x}^{\mathcal{R}}, q_{i,y}^{\mathcal{R}}, q_{i,\theta}^{\mathcal{R}})^T$  and  $q_j^{\mathcal{O}}$  is similarly defined, where the third state parameter corresponds to their heading in  $\mathcal{F}_{\mathcal{W}}$ . Let the set  $\mathcal{X} = \{x_1, \dots, x_u\}$  with time-dependent state  $q^{\mathcal{X}}$  represent a set of potentially mobile targets. Although not always true (e.g., consider a scenario where UAVs are tracking ground vehicles), it is assumed for convenience that  $\mathcal{X} \subseteq \mathcal{O}$  (and, thus,  $q^{\mathcal{X}} \subseteq q^{\mathcal{O}}$ ). Additionally, assume that each member of the team has a maximum angular rate given by  $v_{i,\theta}^{\mathcal{R}} \in \mathbb{R}$ ,  $v_{i,\theta}^{\mathcal{R}} \geq 0$  and augment assumptions **A1–A5** with **A7**  $\forall r_i \in \mathcal{R}$ ,  $r_i$  has a directional sensing modality that captures coverage targets within relative orientation bounds  $-\pi \leq \theta_{i,\min}^{\mathcal{R}} < \theta_{i,\max}^{\mathcal{R}} \leq \pi$  and within fixed-distance,  $\hat{d}_i^{\mathcal{R}} \in \mathbb{R}^+$ . These constraints yield a *local* conical region that shall be denoted  $\mathcal{C}_i^{\mathcal{R}}$

Given these assumptions, Problem 1 can be restated with the additional performance requirement that  $\forall t, \forall x_j \in \mathcal{X}, \exists r_i \in \mathcal{R}, x_j \in \mathcal{C}_i^{\mathcal{R}}$ . In order to address this coverage constraint, we consider the omnidirectional visibility graph of [2] and extend it with the following definition

*Definition 3:* The Directional Visibility Graph is a weighted, simple graph  $G_D(\mathcal{V}_D, \mathcal{E}_D)$  with  $\mathcal{V}_D = \mathcal{R} \cup \mathcal{X}$  and edges given by the set  $\mathcal{E}_D = (\mathcal{R} \cup \mathcal{X}) \times (\mathcal{R} \cup \mathcal{X}) \setminus \{(r_i, r_i) \cup (x_j, x_j) : r_i \in \mathcal{R}, x_j \in \mathcal{X}\}$  with weights

$$f_D(y_i, y_j) = \begin{cases} 1, & (y_i, y_j) \in \mathcal{R} \times \mathcal{R} \\ f_D^{\mathcal{X}}(y_i, y_j), & (y_i, y_j) \in \mathcal{R} \times \mathcal{X} \\ 0, & (y_i, y_j) \in \mathcal{X} \times \mathcal{X} \end{cases} \quad (23)$$

$$f_D^{\mathcal{X}}(r_i, x_j) = \begin{cases} 1, & x_j \in \mathcal{C}_i^{\mathcal{R}} \\ 0, & \text{otherwise} \end{cases} \quad (24)$$

Observe that just like the omnidirectional variation presented in [2], this construct will remain connected if and only if each target is covered by some member of the team.

Adopting the approach outlined in §IV-B, we propose the following definition of  $f_D^{\mathcal{X}}$  as the sigmoidal product

$$f_D^{\mathcal{X}}(q_i^{\mathcal{R}}, q_j^{\mathcal{X}}) = \prod_{i=1}^3 (1 + e^{\gamma_D \Delta_i})^{-1} \quad (25)$$

where  $\Delta_1 \triangleq (\|q_i^{\mathcal{R}} - q_j^{\mathcal{X}}\|_2, -\hat{d}_i^{\mathcal{R}})$ ,  $\Delta_2 \triangleq (\theta_{ij}^{\mathcal{R}} - \theta_{i,\max}^{\mathcal{R}})$ , and  $\Delta_3 \triangleq (\theta_{i,\min}^{\mathcal{R}} - \theta_{ij}^{\mathcal{R}})$  with  $\theta_{ij}^{\mathcal{R}}$  representing the relative orientation of  $x_j$  with respect to  $r_i$ , i.e.,  $\theta_{ij}^{\mathcal{R}} = \text{atan2}(q_{j,y}^{\mathcal{X}} - q_{i,y}^{\mathcal{R}}, q_{j,x}^{\mathcal{X}} - q_{i,x}^{\mathcal{R}}) - q_{i,\theta}^{\mathcal{R}}$  and  $\gamma_D \in \mathbb{R}^+$  a tuning gain. Accordingly, as  $\gamma_D \rightarrow \infty$  it holds that  $f_D^{\mathcal{X}}$  more closely approximates an indicator of  $x_j \in \mathcal{C}_i^{\mathcal{R}}$  (see Figure 1).

Once again, we apply a first-order discretization (see §IV-B) with respect to  $q_i^{\mathcal{R}}$  to yield a discrete representation, which is omitted for the sake of brevity. Given this form,

a discrete-time update for the state-dependent, directional visibility graph laplacian (*i.e.*,  $\mathcal{L}_D$ ) can be formulated, which allows us to enforce visibility over  $\Delta t$  via

$$J_D^T \mathcal{L}_D(k+1) J_D \succ \epsilon_D I_{n+u-1} \quad (26)$$

with  $0 < \epsilon_D \leq n+u = |\mathcal{R}| + |\mathcal{X}|$  representing the desired level of directional visibility to be maintained across the formation and the remaining terms defined similarly to (10).

*Corollary 6.2:* Let  $\mathcal{R}$  and  $\mathcal{O}$  be as previously defined in accordance with **A1 – A5** and **A7** with dynamics (1). Let  $\Gamma: \mathbb{R}^{2n} \rightarrow \mathbb{R}$  and  $\Delta t$  be defined as in Theorem 4.2. Define  $q_{\tau}^{\mathcal{R}} \triangleq (q_x^{\mathcal{R}}, q_y^{\mathcal{R}})^T$ . Solving the following SDP at step  $k$

$$\begin{aligned} \min \quad & \Gamma(q^{\mathcal{R}}(k+1)) + \gamma_s z_s(k+1) \\ \text{s.t.} \quad & |q_{i,\theta}^{\mathcal{R}}(k+1) - q_{i,\theta}^{\mathcal{R}}(k)| \leq v_{i,\theta}^{\mathcal{R}} \Delta t, \quad i = 1, \dots, n \\ & \|q_{i,\tau}^{\mathcal{R}}(k+1) - q_{i,\tau}^{\mathcal{R}}(k)\|_2 \leq v_{i,\tau}^{\mathcal{R}} \Delta t, \quad i = 1, \dots, n \\ & \|A_S q_{\tau}^{\mathcal{R}}(k+1)\|_2 \leq z_s(k+1) \\ & J_P^T \mathcal{L}_P(k+1) J_P = (n+m) I_{n+m-1} \\ & J_N^T \mathcal{L}_N(k+1) J_N \succ \epsilon_N I_{n-1} \\ & J_D^T \mathcal{L}_D(k+1) J_D \succ \epsilon_D I_{n+u-1} \end{aligned} \quad (27)$$

will result in trajectories that minimize  $\Gamma(q^{\mathcal{R}}(k+1)) + \gamma_s z_s(k+1)$  over  $\Delta t$ , while guaranteeing network connectivity, collision-free trajectories, and directional coverage of targets – provided that a feasible solution exists.

## VII. SIMULATION RESULTS

(22) and (27) were implemented in Matlab using SDPT3 as the underlying solver. For the first scenario, we consider the task of driving the team to an objective waypoint  $q^{\mathcal{G}} = (14, -10)^T$ , while achieving a desired ensemble delta formation subject to inter-agent and agent-obstacle proximity constraints, and network connectivity constraints. The orientation of the desired formation (*i.e.*,  $A_S$ ) was fixed during each iteration at  $\theta_S = \text{atan2}(q_y^{\mathcal{G}} - q_{\mu,y}^{\mathcal{R}}(k), q_x^{\mathcal{G}} - q_{\mu,x}^{\mathcal{R}}(k))$ . Three static obstacles occupied the workspace along with two dynamic obstacles. Robot velocity was  $1 \frac{m}{s}$  with dynamic obstacle velocities set at  $0.9 \frac{m}{s}$  and  $0.98 \frac{m}{s}$ . Figure 2 illustrates our results, while Figure 4 illustrates the evolution of the objective terms and Fiedler values.

For the second scenario, we consider a team of four robots charged with tracking a mobile target using a high-resolution directional sensing modality. Here, the pose of the target is chosen to correspond with the translational component of the shape transformation. In contrast to the previous example, the formation orientation and scale were left as open variables.

Figure 3 illustrates these results. The team's objective was to maximize connectivity of the state-dependent directional visibility graph (*i.e.*,  $\Gamma = -\lambda_2(G_D)$ ). The FOV of the sensing modality was chosen as  $\frac{2\pi}{9}$  corresponding to  $\theta_{i,\min}^{\mathcal{R}} = -\frac{\pi}{9}$  and  $\theta_{i,\max}^{\mathcal{R}} = \frac{\pi}{9}$  with sensing range  $\hat{d}_i^{\mathcal{R}} = 5$ . The translational velocity of the pursuers was uniformly chosen as  $1 \frac{m}{s}$  and with  $0.025 \frac{rad}{s}$  as the maximum angular rate.

## VIII. CONCLUSION

In this paper, a discrete-time, optimization-based framework was formulated for multi-robot systems operating

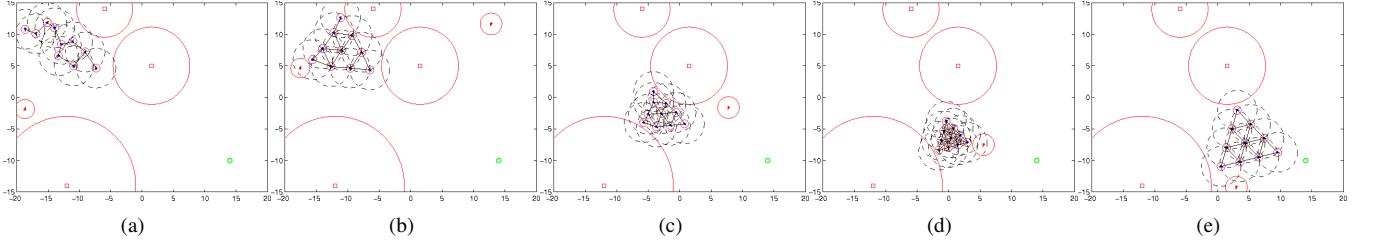


Fig. 2. (a–b) A team of ten robots transitions from an initial inline formation to a delta shape formation and avoids a dynamic obstacle while attempting to maximize formation scale (*i.e.*,  $\alpha_S$ ) and maintain network connectivity. (c) The team deforms shape to preserve inter-agent and static obstacle proximity constraints to safely pass through a passage en-route to goal  $q_1^G = (14, -10)^T$ . (d) The team achieves the desired formation; however, this time at minimal scale to avoid a collision with a second dynamic obstacle. (e) The team obtains the desired shape at maximal scale and achieves the destination waypoint.

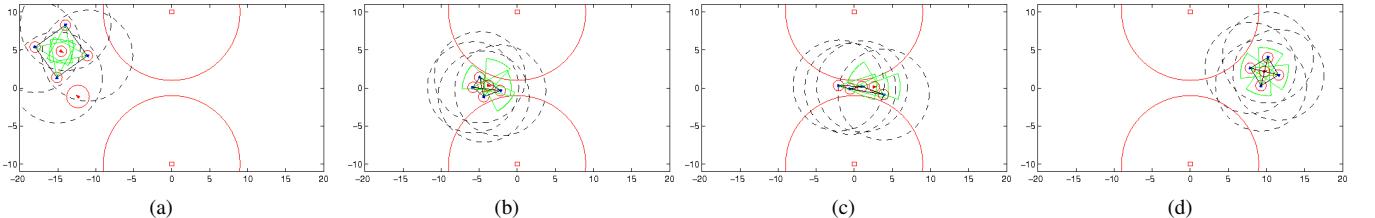


Fig. 3. (a) A team of four networked agents, employing directional sensing modalities, obtain a “bounding box” formation to cage a mobile target. (b–c) After successfully avoiding a mobile obstacle, the team deforms shape to ensure coverage as the evader passes through a passage before again achieving the desired formation at a different orientation and scale.

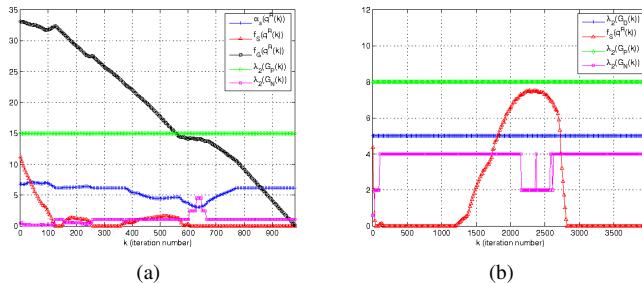


Fig. 4. (a) Evolution of objective terms and state-dependent  $\lambda_2(G_P)$  and  $\lambda_2(G_N)$  corresponding to Figure 2. (b) Evolution of objective terms and  $\lambda_2(G_P)$ ,  $\lambda_2(G_D)$ , and  $\lambda_2(G_N)$  corresponding to Figure 3. Observe that  $\lambda_2(G_P), \lambda_2(G_N) > 0$  for the duration of both runs indicates collision-free trajectories and network connectivity for the teams.

in dynamic environments that embeds both formation and coverage control while guaranteeing inter-agent and agent-obstacle collision avoidance and network connectivity (provided a feasible solution exists). Formation control was embedded via a convex shape-disparity functional that captures the salient features of shape without relaxation. Goal-directed behavior was also illustrated for directional coverage control and waypoint traversal. *It should also be noted that these results readily extend to  $\mathbb{R}^3$  provided the formation shape has a fixed-orientation over  $\Delta t$ .*

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